Singularity free cosmological solutions of Einstein-Maxwell equations

Stoytcho S. Yazadjiev*, Ventseslav A. Rizov †

Department of Theoretical Physics, Faculty of Physics, Sofia University,

5 James Bourchier Boulevard, Sofia 1164, Bulgaria

Abstract

We report on a new two-parameter class of cosmological solutions to the Einstein-Maxwell equations. The solutions have everywhere regular curvature invariants. We prove that the solutions are geodesically complete and globally hyperbolic.

1 Introduction

The reasons to study inhomogeneous cosmologies have deep observational and theoretical bases. As it is well known the present universe is not exactly spacially homogeneous even at large scales. Moreover, there are no reasons to assume that the regular expansion is suitable for a description of the early universe. This strongly motivates the study of inhomogeneous cosmological models. They allows one to investigate a number of long-standing questions regarding the structure formation, the occurrence of singularities, the behavior of the solutions in the vicinity of a singularity and the possibility for our universe to arise from generic initial data. In particular, on the base of some exact solutions it was shown that the nonlinear inhomogeneities could regularize the initial singularity giving rise to completely regular cosmologies both in general relativity [1]-[15] and in alternative gravitational theories [16]-[21]. From a purely theoretical point of view, the investigation of nonsingular cosmological models gives invaluable insight into the spacetime structure, the inherent nonlinear character of gravity and its interaction with matter fields. As a byproduct it also deepens our understanding of the singularity theorems, in particular the assumptions lying in their base [10].

In the present work we present a new two-parameter class of exact singularity free solutions of the Einstein-Maxwell equations.

*E-mail: yazad@phys.uni-sofia.bg

†E-mail: rizov@phys.uni-sofia.bg

2 The exact solutions

We consider Einstein-Maxwell gravity described by the equations:

$$
\begin{aligned}
R_{\mu\nu} = 2F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},\\
\nabla_{\mu}F^{\mu\nu} = 0,\\
\nabla_{\alpha}F_{\mu\nu} + \nabla_{\nu}F_{\alpha\mu} + \nabla_{\mu}F_{\nu\alpha} = 0.
\end{aligned}
\tag{1}
$$

We assume that the spacetime admits two spacelike commuting Killing vectors $\partial/\partial\phi$ and $\partial/\partial z$, which are mutually- and hypersurface-orthogonal. The metric then can be written in the well known form [22]:

$$
ds^2 = e^{2\gamma(t,r)}(-dt^2 + dr^2) + e^{2\psi(t,r)}\rho^2(t,r)d\phi^2 + e^{-2\psi(t,r)}dz^2. \tag{2}
$$

We have found the following two-parameter class of exact solutions:

$$
\begin{aligned}
e^{\psi(t,r)} &= \frac{\lambda^2 + (1-\lambda^2)\cosh^4(ar)\cosh^2(2at)}{\cosh^2(ar)\cosh(2at)},\\
\rho(t,r) &= \frac{1}{a}\cosh(ar)\sinh(ar)\cosh(2at),\\
e^{\gamma(t,r)} &= \cosh^2(ar)\left[\lambda^2 + (1-\lambda^2)\cosh^4(ar)\cosh^2(2at)\right],\\
F_{03} &= 4a\lambda\sqrt{1-\lambda^2}e^{-2\psi(t,r)}\tanh(2at),\\
F_{13} &= 4a\lambda\sqrt{1-\lambda^2}e^{-2\psi(t,r)}\tanh(ar),
\end{aligned}
\tag{3}
$$

where $a > 0$ and $0 < \lambda < 1$ are free parameters. The limiting cases $\lambda = 0$ and $\lambda = 1$ correspond to solutions of the vacuum Einstein equations. The limit $a \to 0$ gives the Minkowski spacetime. The range of the coordinates is:

$$
-\infty < t, z < \infty \ \ , 0 < r < \infty, \ \ 0 \le \phi < 2\pi. \tag{4}
$$

The spacetimes described by the solutions (3) have well defined axis of symmetry [22]. Therefore, it can be said that these spacetimes admit cylindrical symmetry.

It should be noted that new solutions with the same spacetime geometry and different Maxwell tensor can be obtained by a duality rotation

$$
F_{\mu\nu} \to F_{\mu\nu}\cos(\theta) + \star F_{\mu\nu}\sin(\theta), \tag{5}
$$

where θ is a constant parameter and $\star$ is the Hodge dual.

3 Curvature invariants

The components of the Weyl tensor in the standard null tetrad are:

$$\begin{aligned}
e^{2\gamma(t,r)}\Psi_0 = & \frac{3a^2}{\cosh^2(ar)} - 2a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\frac{2}{\cosh^2(2at)}+\frac{1}{\cosh^2(ar)}\right] \\
& -\frac{8a^2\lambda^2}{S(t,r)}\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\tanh^2(2at)+\tanh^2(ar)\right] \\
& +16a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]^2\left[\tanh(2at)+\tanh(ar)\right]^2 \\
& -8a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\tanh(2at)+\tanh(ar)\right]\tanh(2at)
\end{aligned} \tag{6}$$

$$\Psi_1 = 0, \tag{7}$$

$$3e^{2\gamma(t,r)}\Psi_2 = \frac{3a^2}{\cosh^2(ar)} + 2a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\frac{1}{\cosh^2(ar)}-\frac{2}{\cosh^2(2at)}\right]$$

$$+\frac{8a^2\lambda^2}{S(t,r)}\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\tanh^2(ar)-\tanh^2(2at)\right] \tag{8}$$

$$+16a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]^2\left[\tanh^2(2at)-\tanh^2(ar)\right] \tag{9}$$

$$+4a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[3\tanh^2(ar)-2\tanh^2(2at)-1\right] \tag{10}$$

$$\Psi_3 = 0, \tag{11}$$

$$\begin{aligned}
e^{2\gamma(t,r)}\Psi_4 = & \frac{3a^2}{\cosh^2(ar)} - 2a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\frac{2}{\cosh^2(2at)}+\frac{1}{\cosh^2(ar)}\right] \\
& -\frac{8a^2\lambda^2}{S(t,r)}\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\tanh^2(2at)+\tanh^2(ar)\right] \\
& +16a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]^2\left[\tanh(2at)-\tanh(ar)\right]^2 \\
& -8a^2\left[1-\frac{\lambda^2}{S(t,r)}\right]\left[\tanh(2at)-\tanh(ar)\right]\tanh(2at)
\end{aligned} \tag{12}$$

where $S(t,r) = \lambda^2 + (1-\lambda^2)\cosh^4(ar)\sinh^2(2at)$.

Further, the components of the Maxwell tensor in the standard null tetrad read

$$\Phi_0 = \frac{-i}{S(t,r)} 2a\lambda\sqrt{1-\lambda^2}\cosh(2at)[\tanh(2at) - \tanh(ar)], \tag{13}$$

$$\Phi_1 = 0, \tag{14}$$

$$\Phi_2 = \frac{i}{S(t,r)} 2a\lambda\sqrt{1-\lambda^2})\cosh(2at)[\tanh(2at) + \tanh(ar)]. \tag{15}$$

All components of the Weyl tensor and the Maxwell tensor are regular everywhere. Since the scalar invariants that can be formed with the metric and the Riemann curvature tensor are polynomials of these components, all curvature invariants are regular everywhere. From the explicit form of the Weyl tensor components one can check that the spacetimes are of Petrov type I except at the axis where they are of type D.

4 Geodesic completeness

In order to demonstrate the geodesic completeness of the above solutions we have to show that all causal geodesics can be extended to arbitrary values of the affine parameter. Since the metric functions are even in the time variable we shall investigate only future-directed geodesics. Below we consider $0 < \lambda < 1$.

The existence of isometries gives rise to two constants of motion along the geodesics:

$$\begin{aligned} L &= e^{2\psi(t,r)}\rho^2(t,r)\frac{d\phi}{d\tau}, \\ P &= e^{-2\psi(t,r)}\frac{dz}{d\tau}, \end{aligned} \tag{16}$$

where we have denoted by τ the affine parameter along the geodesics.

The affinely parameterized geodesics satisfy

$$e^{2\gamma(t,r)}\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2\right] - \frac{L^2}{\rho^2(t,r)}e^{-2\psi(t,r)} - P^2 e^{2\psi(t,r)} = \epsilon, \tag{17}$$

where $\epsilon = 1$ and $\epsilon = 0$ for timelike and null geodesics, respectively. Writing $d\phi/d\tau$ and $dz/d\tau$ as functions of L and P, the geodesic equations for t and r can be written in the following form [14],[15]:

$$\frac{d}{d\tau}\left(e^{2\gamma(t,r)}\frac{dt}{d\tau}\right) = e^{-2\gamma(t,r)}M(t,r)\partial_t M(t,r), \tag{18}$$

$$\frac{d}{d\tau}\left(e^{2\gamma(t,r)}\frac{dr}{d\tau}\right) = -e^{-2\gamma(t,r)}M(t,r)\partial_r M(t,r), \tag{19}$$

where the function $M(t,r)$ is defined by

$$M(t,r) = e^{\gamma(r,r)}\left[\epsilon + \frac{L^2}{\rho^2(t,r)}e^{-2\psi(t,r)} + P^2 e^{2\psi(t,r)}\right]^{1/2}. \tag{20}$$

First we consider the null geodesics with $L = P = 0$. For this case we have $dt/d\tau = |dr/d\tau|$ and

$$\frac{d}{d\tau}\left(e^{2\gamma(t,r)}\frac{dt}{d\tau}\right) = 0. \tag{21}$$

After integration, we obtain

$$\frac{dt}{d\tau} = Ce^{-2\gamma(t,r)}, \tag{22}$$

where $C > 0$ is a constant. One can easily see from the explicit form of $e^{\gamma(t,r)}$ that $e^{-2\gamma(t,r)} \leq 1$. As a consequence one finds

$$\frac{dt}{d\tau} = |\frac{dr}{d\tau}| \leq C \tag{23}$$

and therefore the null geodesics with $L = P = 0$ are complete.

Let us turn to the general case when at least one of the constants ϵ, L or P is different from zero. In this case we parameterize $dt/d\tau$ and $dr/d\tau$ by writing [14],[15]:

$$\begin{aligned}\frac{dt}{d\tau} &= e^{-2\gamma(t,r)}M(t,r)\cosh(\upsilon),\\ \frac{dr}{d\tau} &= e^{-2\gamma(t,r)}M(t,r)\sinh(\upsilon).\end{aligned} \tag{24}$$

Substituting these expressions into the equations for $dt/d\tau$ and $dr/d\tau$, we obtain the following equation for υ:

$$\frac{d\upsilon}{d\tau} = -e^{-2\gamma(t,r)}\left[\partial_t M(t,r)\sinh(\upsilon) + \partial_r M(t,r)\cosh(\upsilon)\right]. \tag{25}$$

The above equation can be written in a more explicit form

$$\frac{d\upsilon}{d\tau} = -\frac{1}{M(t,r)}\{J_+(t,r)\cosh(\upsilon) + J_-(t,r)\sinh(\upsilon)\}, \tag{26}$$

where

$$\begin{aligned}J_+(t,r) = 2a\epsilon\tanh(ar)&\left[1 + \frac{2(1-\lambda^2)\cosh^4(ar)\cosh^2(2at)}{\lambda^2 + (1-\lambda^2)\cosh^4(ar)\cosh^2(2at)}\right]\\ &+\frac{a^3L^2[3\tanh^2(ar) - 1]}{\tanh^3(ar)[\lambda^2 + (1-\lambda^2)\cosh^4(ar)\cosh^2(2at)]}\\ +8aP^2(1-\lambda^2)&[\lambda^2 + (1-\lambda^2)\cosh^4(ar)\cosh^2(2at)]\tanh(ar),\end{aligned} \tag{27}$$

$$
\begin{aligned}
J_-(t,r) = 4a\epsilon &\frac{(1-\lambda^2)\cosh^4(ar)\cosh^2(2at)}{\lambda^2+(1-\lambda^2)\cosh^4(ar)\cosh^2(2at)} \\
+2aP^2 &\frac{[\lambda^2+(1-\lambda^2)\cosh^4(ar)\cosh^2(2at)]}{\cosh^4(ar)\cosh^2(2at)} \\
\times &\left[3(1-\lambda^2)\cosh^4(ar)\cosh^2(2at)-\lambda^2\right]\tanh(2at).
\end{aligned}
\tag{28}
$$

In order for the geodesics to be complete, the functions $dt/d\tau$ and $dr/d\tau$ have to remain finite for finite values of the affine parameter. In fact, it is sufficient to consider only $dt/d\tau$ since $dt/d\tau$ and $dr/d\tau$ are related via (17) and $dr/d\tau$ cannot become singular if $dt/d\tau$ is not singular. The derivatives $d\phi/d\tau$ and $dz/d\tau$ are regular functions of t and r and cannot become singular if $t(\tau)$ and $r(\tau)$ are not singular. The only problem we could have is when $r(\tau)$ approaches the value $r=0$ for $L\neq 0$. We shall show, however, that $r(\tau)$ cannot become zero for $L\neq 0$.

First we consider the geodesics with increasing r (i.e., $\upsilon>0$). From the explicit form of the functions $e^{2\gamma(t,r)}$ and $M(t,r)$ it is not difficult to see that

$$
e^{-2\gamma(t,r)}M(t,r) \leq \left[\epsilon + P^2 + \frac{a^2L^2}{\sinh^2(ar)}\right]^{1/2}. \tag{29}
$$

Therefore, $dt/d\tau$ could become singular only because of $\upsilon(\tau)$. However, for increasing r, $\upsilon(\tau)$ cannot diverge since for large t (large r) the derivative $dt/d\tau$ becomes negative as can be seen from (26), (27) and (28).

Let us now consider the second case when $r(\tau)$ decreases ($\upsilon<0$). The geodesics with $L=0$ reach the axis $r=0$ smoothly and then reappear with $dr/d\tau>0$ ($\upsilon>0$). A problem may arise from $r=0$ for $L\neq 0$. However, for $L\neq 0$, $r(\tau)$ cannot become zero and this can be shown as follows. When $r(\tau)$ approaches zero the dominant term is the one associated with L and the other terms can be ignored. So, for very small r the geodesics behave as null geodesics with $P=0$:

$$
\begin{aligned}
\frac{dt}{d\tau} &= e^{-2\gamma(t,r)}M(r)\cosh(\upsilon), \\
\frac{dr}{d\tau} &= e^{-2\gamma(t,r)}M(r)\sinh(\upsilon), \\
\frac{d\upsilon}{d\tau} &= -e^{-2\gamma(t,r)}\partial_r M(r)\cosh(\upsilon),
\end{aligned}
\tag{30}
$$

where

$$
M(r) = |aL|\frac{\cosh^3(ar)}{\sinh(ar)}. \tag{31}
$$

Hence, one finds

$$
\frac{dr}{d\upsilon} = -\frac{M(r)}{\partial_r M(r)}\tanh(\upsilon). \tag{32}
$$

After integration we have

$$\sinh(ar) = D\cosh^3(ar)\cosh(\upsilon), \tag{33}$$

where $D > 0$ is a constant. From here one can immediately see that $r(\tau)$ cannot become zero. So, we have proven that our solutions are geodesically complete. In the same manner it can be shown that the vacuum solutions for $\lambda = 0$ and $\lambda = 1$ are geodesically complete, too.

From the above considerations it follows that every maximally extended null geodesic intersects once and only once any of the hypersurfaces $t = constant$. Therefore the hypersurfaces $t = constant$ are global Cauchy surfaces [23] and the spacetimes described by the solutions are globally hyperbolic.

Finally, it is interesting to see which of the assumptions of the singularity theorems [24] turn out to be violated. Since the energy and the causal conditions are fulfilled it remains to conclude that the spacetimes considered do not contain a closed trapped surface. In order to prove this we shall follow the considerations of [2],[25] and [26]. Indeed, let us suppose that the there is one such surface. Then, since the surface is compact, there must be a point q where r reaches its maximum. Let us denote it by $r_{max} = R$ on a constant time hypersurface $t = T$. For the traces of both null second fundamental forms at q, it can be shown that

$$\begin{aligned} K^{+} &\geq \sqrt{2}ae^{-\gamma(R,T)}\frac{1+\tanh(2at)\tanh(2ar)}{\tanh(2ar)} > 0, \\ K^{-} &\leq \sqrt{2}ae^{-\gamma(R,T)}\frac{\tanh(2at)\tanh(2ar)-1}{\tanh(2ar)} < 0. \end{aligned} \tag{34}$$

The traces have opposite sings, and, therefore, there are no closed trapped surfaces.

In conclusion, we have presented a new (to the best of our knowledge) two-parameter class of exact solutions of the Einstein-Maxwell equations. The solutions have no curvature singularity. Moreover, they are geodesically complete and globally hyperbolic. The solutions can be viewed as explicit examples of how the nonlinear inhomogeneities can regularize the singularities yielding completely regular spacetimes.

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